HITACHI Analog•Hybrid Computer

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Partial Differential Equations for Heat Conduction Analysis of Frozen Layer Shifting.

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Partial Differential Equations for Heat Conduction Analysis of Frozen Layer Shifting

In analyzing partial differential equations for heat conduction on the analog computer, the boundary face between two different substances has been usually assumed not to shift. In the present example, however, since temperature distribution in the process of freezing is expressed as a function of time, the boundary face between frozen and non-frozen layer is unintentionally displaced with time, presenting a very interesting way of analysis by the analog computer.

1. Physical Conditions

In the process of freezing, in which the frozen layer proceeds with time, shifting of the boundary face between frozen and non-frozen layers occurs, and the position of boundary face and temperature distribution are sought for as a function of time. In this case, heat conduction is assumed to occur unidirectionally.

$$\frac{\partial t_1}{\partial \theta} = K_1 \frac{\partial^2 t_1}{\partial x^2} \qquad (o < x < \delta) \qquad (1)$$

$$\frac{\partial t_2}{\partial \theta} = K_2 \frac{\partial^2 t_2}{\partial x^2} \qquad (\delta < x) \qquad (2)$$

Initial conditions

For $\theta = o$, $t_1 = t_2 = C_1$ (3)

For $\theta = o$, $\delta = o$ (4)

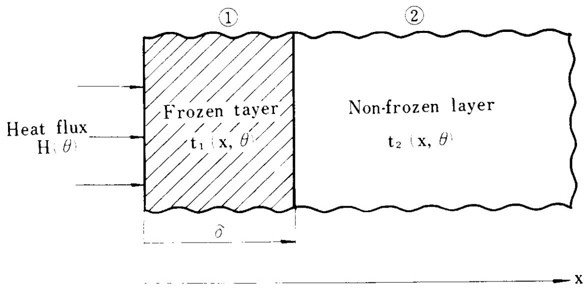

Fig. 1 Physical Phenomenon

Boundary conditions

At $x = o$ $\quad H(\theta) = -\lambda_1 \frac{\partial t}{\partial x} = h(t_a - t_1)$ (5)

$x = \delta$ $\quad \lambda_1 \frac{\partial t_1}{\partial x} - \lambda_2 \frac{\partial t_2}{\partial x} = \rho \gamma \frac{\partial \delta}{\partial \theta}$ (6)

$x = \delta$ $\quad t_1 = t_2 = o$ (7)

$x = L$ $\quad t_2 = C_1$ (8)

where λ_1	:	thermal conductivity of ice	1.05 Kcal/m hr °C
λ_2	:	thermal conductivity of water	0.344 Kcal/m hr°C
Cp_1	:	specific heat of ice	0.475 Kcal/kg °C
Cp_2	:	specific heat of water	0.889 Kcal/kg °C
ρ_1	:	density of ice	980 kg/m³
ρ_2	:	density of water	1,020 kg/m³
C_1	:	initial temperature	+3.9°C
t_a	:	atmospheric temperature	-20°C
h	:	thermal conduction coefficient of air	5.5 Kcal/m² hr °C
γ_1	:	heat of fusion	65.2 Kcal/kg
γ_2	:	heat of fusion	80.0 Kcal/kg
θ	:	time	hr
t_1	:	temperature of frozen layer	°C
t_2	:	temperature of non-frozen layer	°C
δ	:	distance from the surface of frozen layer to the boundary face	m
K_1	:	thermal diffusibility of ice	
K_2	:	thermal diffusibility of water	
Q	:	heat density	Kcal/m³
x	:	distance from the surface of frozen layer	m

The freezing conditions are assumed as follows: the medium is in contact with the atmosphere of temperature -20°C and infinitely large heat capacity, and convection in the water phase is neglected.

2. Conversion

In the equation of heat conduction

$$\frac{\partial t}{\partial \theta} = K \frac{\partial^2 t}{\partial x^2} \tag{9}$$

putting $$K = \frac{\lambda}{\rho C p} \tag{10}$$

$$\rho C p \frac{\partial t}{\partial \theta} = \lambda \frac{\partial^2 t}{\partial x^2} \tag{11}$$

Let us consider freezing process in unit area.

Heat contained in unit volume is

$$Q = \rho C p t \tag{12}$$

Putting (12) into (11),

$$\frac{\partial \theta}{\partial \theta} = \frac{\partial^2 (\lambda t)}{\partial x^2} \tag{13}$$

Assume that the medium to be frozen, with thickness L, is divided into n parts, and the average temperature of each part is t_{11}, t_{12}, t_{13} ······························· $t_1 n^{\circ}$
If each part has volume dV (= dx × 1 m²), heat Q_n contained in dv is

$$Q n = \rho_2 C p_2 t_{1n} \; d v \tag{14}$$

if the part is pure water and

$$Q n = \rho_1 C p_1 t_{1n} \; d v - 65.2 (\text{ or } 80.0) \times d v, \tag{15}$$

if the part is of pure ice.

In order to avoid complication in calculation, volume increase due to freezing is neglected, and heat contained in dV of water at 0°C is regarded to be 0.

From Eqs. (14) and (15), λt is plotted against Q as shown in Fig. 2

If Eq. (13) is converted to a difference equation,

$$\frac{d Q i}{d \theta} = \frac{1}{(d x)^2} \{ (\lambda t)_{i+1} - 2 (\lambda t)_i + (\lambda t)_{i-1} \} \tag{16}$$

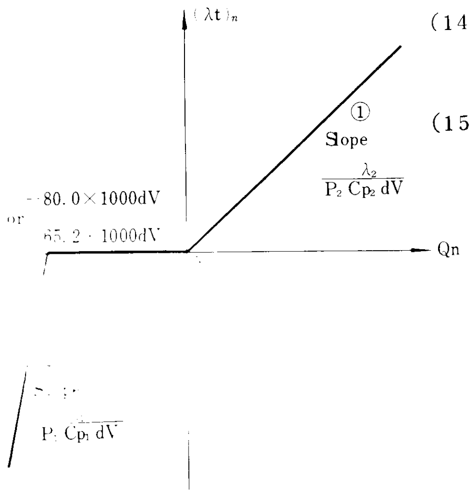

Fig. 2 The relationship between Qn and (λt)n

Since $|t| \leq 20$℃, so $|(\lambda t)|$ max $< 20 \times 1.05$.
Taking a favorable scale factor, the computer variable is set to $\left[\frac{(\lambda t) i}{40}\right]$

It is easily understood that the maximum of $|Qi|$ does not exceed 10^4 Kcal/dV if the width of the partition range is set to 0.1 m.

Hence, the equation after scaling becomes as in (17).

$$\left[\frac{1}{10^4} \frac{dQi}{d\theta}\right] = \frac{40}{10^4 (d x^2)} \left\{ \left[\frac{(\lambda t)_{i+1}}{40} - 2 \frac{(\lambda t)_i}{40} + \frac{(\lambda t)_{i-1}}{40} \right] \right\} \tag{17}$$

In order to obtain a representation compatible with the machine units of the computer, the graph of Fig. 2 is changed to that of Fig. 3.

The boundary condition is, at x = 0,

$$\left(\frac{\partial t}{\partial x}\right)_{x=0} \fallingdotseq \frac{t_{12} - t_{11}}{dx} \qquad (18)$$

Putting (18) into (5),

$$-\lambda_1 \frac{t_{12} - t_{11}}{dx} = h\,(t_a - t_{11}) \qquad (19)$$

or

$$t_{11} = \frac{h\,t_a + \lambda_1\, t_2 / dx}{\lambda_1 / dx + h} \qquad (20)$$

Accordingly,

$$\frac{(\lambda t)_1}{40} \fallingdotseq \frac{1}{40} \times \lambda_1 \times t_{11} \fallingdotseq$$

$$\frac{\lambda_1 h t_a}{40(\lambda_1 / dx + h)} + \frac{\lambda_1 / dx}{(\lambda_1 / dx + h)} + \frac{(\lambda t)_2}{40}$$

$$\therefore \frac{(\lambda t)_1}{40} \fallingdotseq \frac{-5.775}{32} + \frac{10.5}{16} \times \frac{(\lambda t)_2}{40} \qquad (21)$$

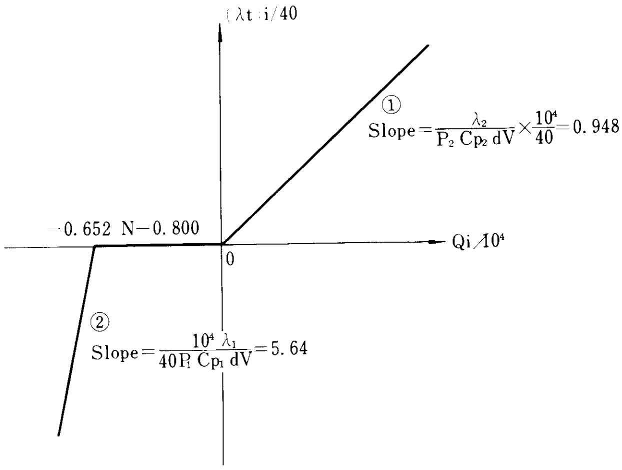

Fig. 3 The relationship between Qn and $(\lambda t)n$ in accordance with machine unit representation ($dV = 0.1\ m^3$)

3. Operation of Analog Computer

Although the accuracy is much improved by dividing the range as finely as possible in handling such difference equations as mentioned in the above, in the present experiment, a water depth of 0.8 m is equally divided into 8 steps at 0.1 m intervals, approximating the total range with 7 difference equations and an algebraic equation.

There are two ways of setting boundary conditions at x = Lm,

$$t = \text{const.}$$

$$\text{and} \quad \frac{\partial t}{\partial x} = 0$$

The latter means that a perfectly insulating wall is placed at x = L, which may be a good approximation if the wall is regarded as constituting a part of the refrigerating chamber. However, the former has more practical significance as an approximation for infinite water depth. Anyway, the more the range is divided, the less error between different ways of approximation. In the present report, attention was focused on how the solution waveform is affected by the way of setting boundary conditions Equations.

$$\frac{(\lambda t)_1}{40} = -\frac{5.775}{32} + \frac{10.5}{16} \times \frac{(\lambda t)_2}{40} \qquad (22)$$

$$\frac{1}{10^4}\frac{dQ_2}{d\theta} = \frac{40}{10^4(dx)^2}\left\{\frac{(\lambda t)_3}{40} - 2\frac{(\lambda t)_2}{40} + \frac{(\lambda t)_1}{40}\right\} \qquad (23)$$

$$\frac{1}{10^4}\frac{dQ_3}{d\theta} = \frac{40}{10^4(dx)^2}\left\{\frac{(\lambda t)_4}{40} - 2\frac{(\lambda t)_3}{40} + \frac{(\lambda t)_2}{40}\right\} \qquad (24)$$

$$\frac{1}{10^4}\frac{dQ_4}{d\theta} = \frac{40}{10^4(dx)^2}\left\{\frac{(\lambda t)_5}{40} - 2\frac{(\lambda t)_4}{40} + \frac{(\lambda t)_3}{40}\right\} \qquad (25)$$

$$\frac{1}{10^4}\frac{dQ_5}{d\theta} = \frac{40}{10^4(dx)^2}\left\{\frac{(\lambda t)_6}{40} - 2\frac{(\lambda t)_5}{40} + \frac{(\lambda t)_4}{40}\right\} \qquad (26)$$

$$\frac{1}{10^4}\frac{dQ_6}{d\theta} = \frac{40}{10^4(dx)^2}\left\{\frac{(\lambda t)_7}{40} - 2\frac{(\lambda t)_6}{40} + \frac{(\lambda t)_5}{40}\right\} \qquad (27)$$

$$\frac{1}{10^4}\frac{dQ_7}{d\theta} = \frac{40}{10^4(dx)^2}\left\{\frac{(\lambda t)_8}{40} - 2\frac{(\lambda t)_7}{40} + \frac{(\lambda t)_6}{40}\right\} \qquad (28)$$

$$\frac{1}{10^4}\frac{dQ_8}{d\theta} = \frac{40}{10^4(dx)^2}\left\{\frac{(\lambda t)_9}{40} - 2\frac{(\lambda t)_8}{40} + \frac{(\lambda t)_7}{40}\right\} \qquad (29)$$

The last equation should be replaced for the boundary condition $\left(\frac{\partial t}{\partial x}\right)_{x=L} = 0$ with

$$\frac{1}{10^4}\frac{dQ_8}{d\theta} = \frac{40}{10^4(dx)^2}\left\{\frac{(\lambda t)_9}{40} - \frac{(\lambda t)_8}{40}\right\} \tag{29'}$$

$$\frac{(\lambda t)_9}{40} = \frac{1}{40}\lambda_2 \times t_{19} = \frac{1}{40} \times 0.344 \times 3.9 = 0.0336 \tag{30}$$

$(\lambda t)\ i = (Qi)$ (The function form is shown in Fig. 3.) (31)

The results obtained by analyzing these equations are illustrated in Figs. 4 – 7. It should be noted that considerable differences occur depending upon the boundary conditions. As is clear from the fact that the boundary conditions affect the solution less at the vicinity of the surface, it may be supposed that the more the range is divided, the less error due to difference in boundary conditions.

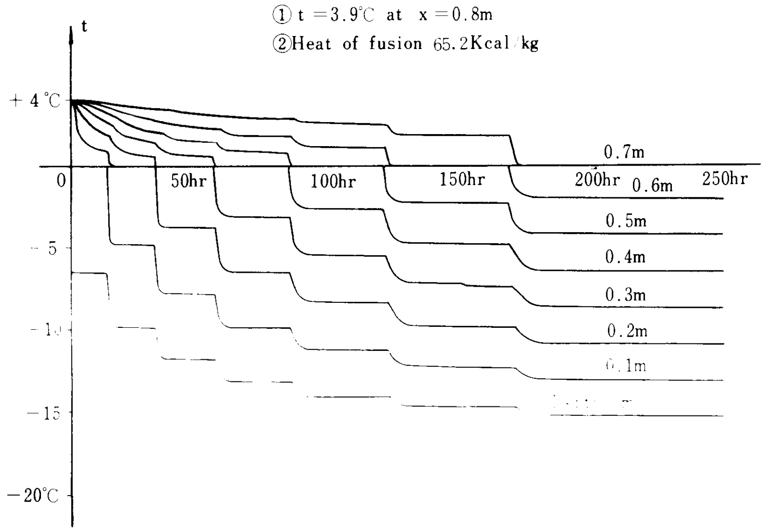

Fig. 4 Temperature transition with depth as parameter

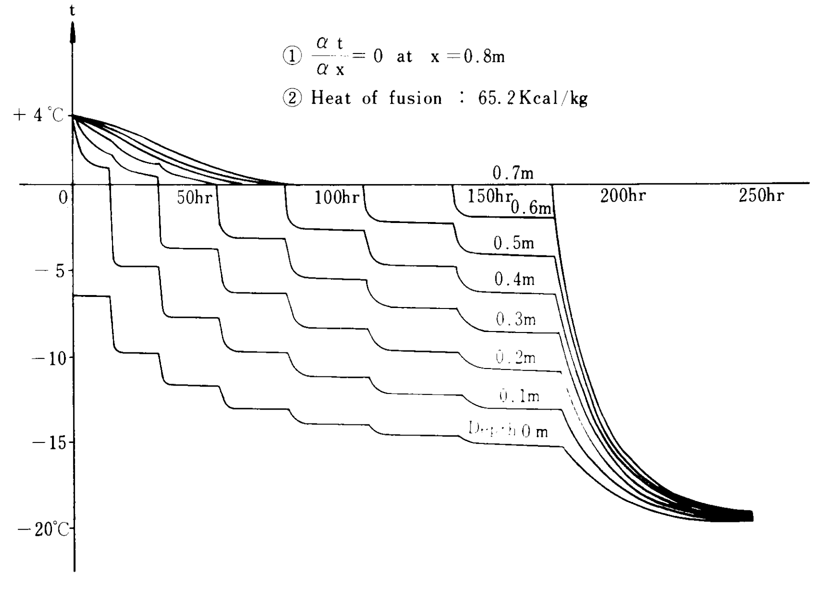

Fig. 5 Temperature transition with depth as parameter

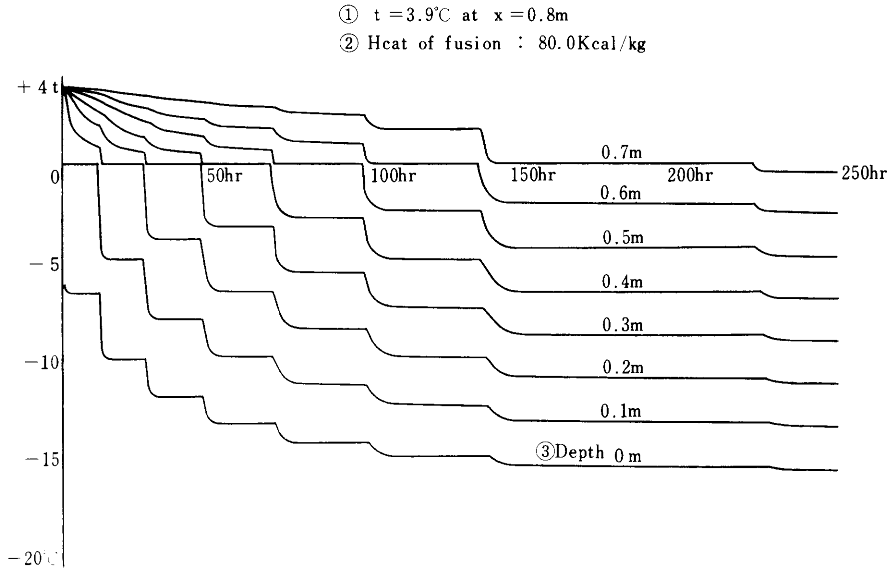

Fig. 6 Temperature transition with depth as parameter

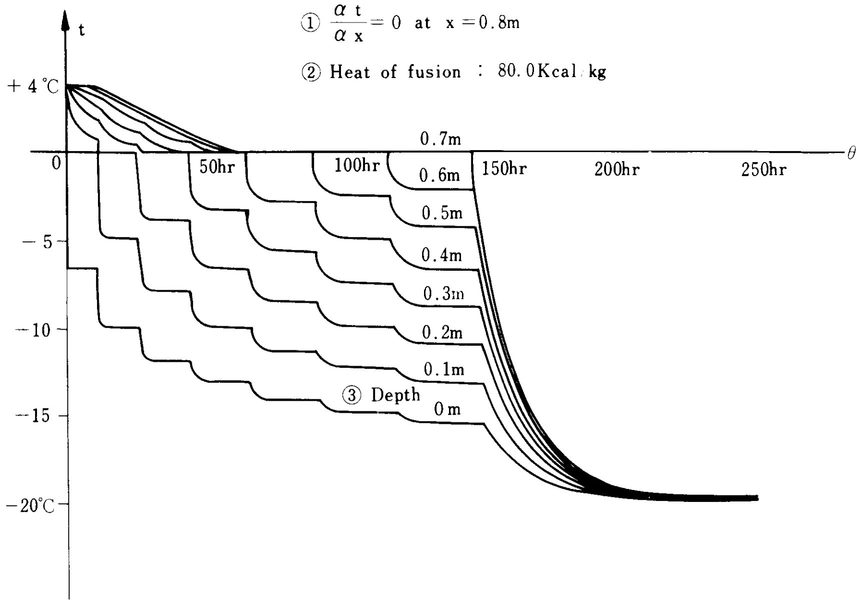

Fig. 7 Temperature transition with depth as parameter

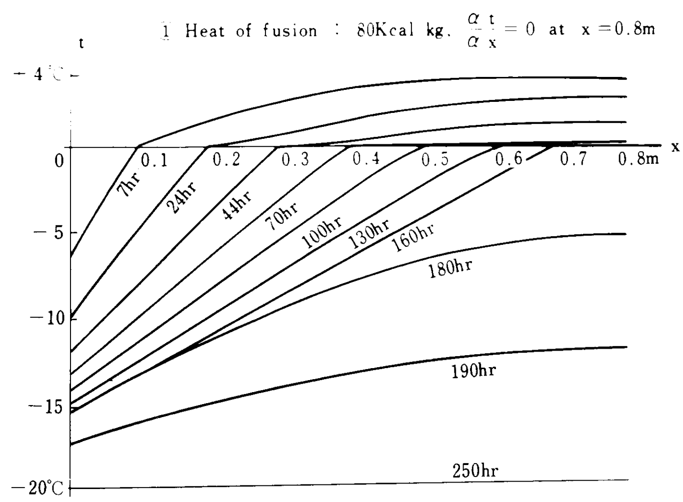

Fig. 8 Temperature distribution with time as parameter

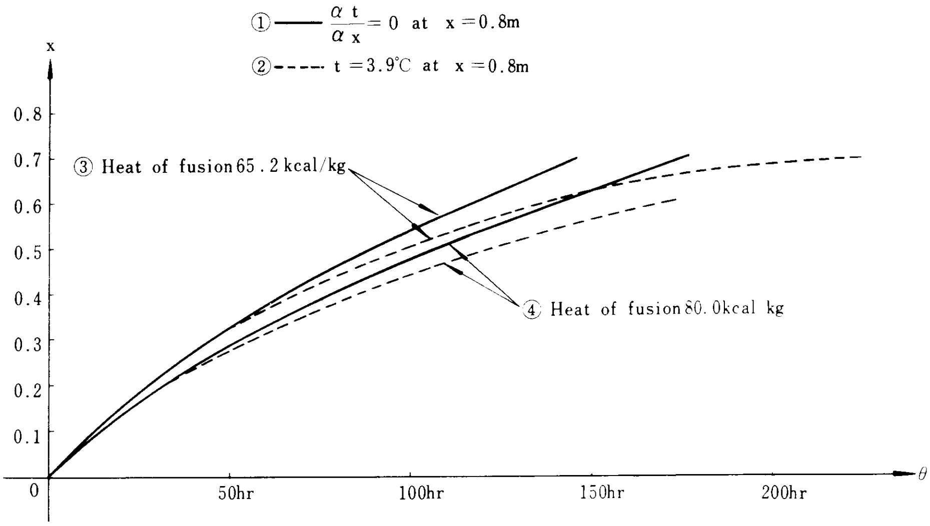

Fig. 9 Displacement of boundary face

4. Discussion

(1) In the present experiment, the point of 0.8 m depth was always held either at a temperature of 3.9°C or at temperature gradient 0, for computing temperature changes at points of 0 ~ 0.7 m depth. These data, however, are not restricted to the 0.8 m boundary, depending upon time scale keeping operation. For instance, temperature transition at 1 m depth obtained by setting 0.8 m depth at 3.9°C can be read as that at 1 m depth obtained by keeping 8 m depth at 3.9°C, if the time axis is compressed 1/10 times.

(2) Since the data presented here are obtained by approximation with division-in-8, the solution waveforms take a stepwise transition, differing considerably from those expected on the basis of physical phenomena. This is inevitable so long as approximation is made by difference equations, and in order to obtain smooth curves, approximation should be improved by increasing the number of divisions. However, when finer division is [illegible] operational elements, the error may be augmented contrarily unless sufficient care is taken in setting potentiometers. It seems desirable to [illegible] of divisions as large as the computer capacity permits, irrespective of operational errors, when the partial differential equations are analyzed.

(3) The displacement of the boundary between ice and water is plotted as a function of time in Fig. 9, in which the curves present some disaccord depending upon the boundary conditions approximating infinite depth, while they agree fairly well with each other in time shorter than 50 hrs. This is also ascribed to the problem of division number stated in (2): the larger the number of divisions, the less error due to boundary condition setting. In this analysis, displacement of boundary layers within the same section is not taken into consideration.

This computation was executed in response to a request from the Engineering Faculty, Kanazawa University. The model employed was HITACHI 505 High Precision Analog Computer. If potentiometers are saved as much as possible, the necessary composition is as follows:

Integrating amplifier	7
Summing amplifier	9
Sign changer	16
Potentiometer	20
Dead zone unit	7
Diode	16

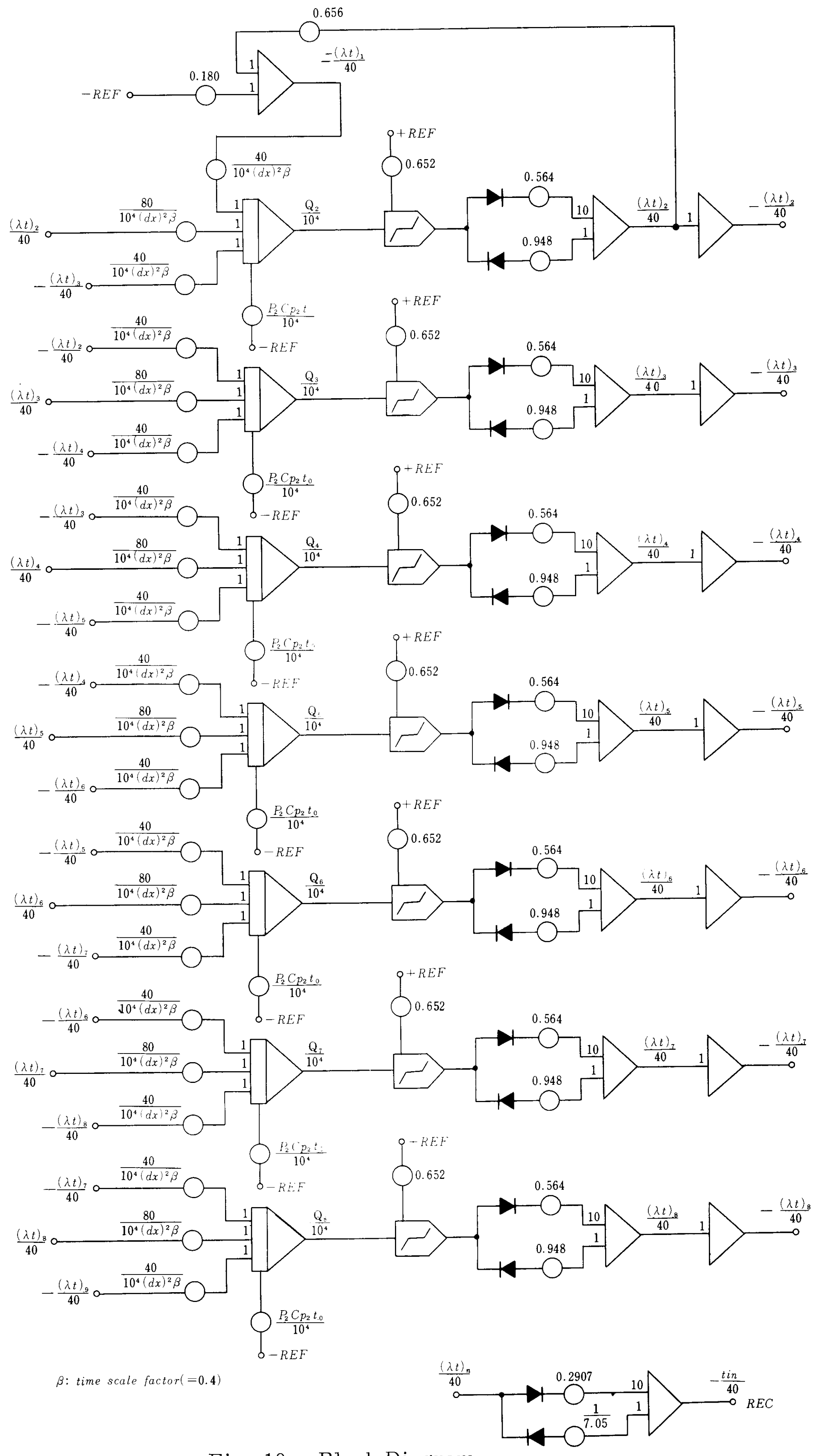

Fig. 10 Block Diagram

β : time scale factor (=0.4)