GIT/EES Report A588/P1

12 January 1962

# DEVELOPMENT OF NEW METHODS AND APPLICATIONS OF ANALOG COMPUTATION

I - An Experimental Electronic Generalized Integrator

II - Nonstationary Noise for Monte Carlo Studies




by

R. S. Johnson

F. R. Williamson

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QUARTERLY PROGRAM REPORT

on

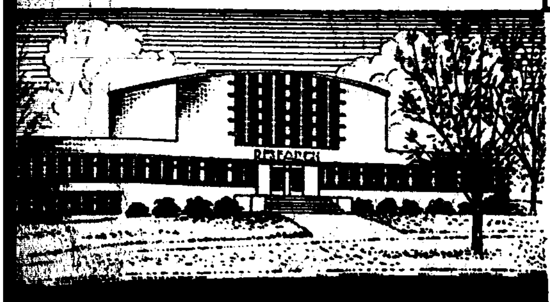
Contract No. NAS8-2473



12 September 1961 - 12 December 1961

For

GEORGE C. MARSHALL SPACE FLIGHT CENTER
Huntsville, Alabama




Engineering Experiment Station

Georgia Institute of Technology

Atlanta, Georgia

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Submitted:

Robert S. Johnson
Project Director A-588

Approved:

F. Dixon, Head
Special Problems Group
Physical Sciences Division

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Chapter I

# BACKGROUND INFORMATION

## 1-1. Introduction

Georgia Tech Research Project No. A-588 was established on 12 September 1961 to assist the Flight Simulation Branch of the Computation Division at the George C. Marshall Space Flight Center (MSFC) in the investigation and development of new methods and applications of analog computation within the following areas of interest:

A - Development of a Generalized Integrator

B - Analog-Computer Statistical Analysis

C - Analog-Computer Partial Differential Equation Solution

This project has been assigned to the Georgia Tech Analog Computer Laboratory (ACL), which operates under the Special Problems Group of the Engineering Experiment Station's Physical Sciences Division.

## 1-2. Project Objectives

In accordance with decisions reached during the project organizational meeting with the Contracting Officer's Representative (Dr. W.P. Krause of MSFC), primary emphasis has been placed on areas A and B mentioned above. It was agreed that the following specific assignments would be undertaken initially:

Task I. Investigation of electronic techniques for integrating analog voltage signals with respect to arbitrary variables. The task includes construction of a working model of a generalized integrator suitable for use with existing analog equipment.

Task II. Investigation of analog techniques for the generation of nonstationary noise voltages to be used in analog Monte Carlo studies. Of particular interest is the problem of nonstationary shaping of Gaussian, band-limited, white noise.

## Chapter II
## ELECTRONIC GENERALIZED INTEGRATOR

### 2-1. Introduction

The ability to perform integration with respect to an arbitrary variable can increase the flexibility of the general-purpose analog computer. We may distinguish the process of integration with respect to variables other than time as "generalized integration." An all-electronic device is being developed under this project to perform such integration. It will be referred to herein as the Electronic Generalized Integrator. Electromechanical devices designed for the same purpose, but with more limited bandwidth, have been previously described in the literature.

### 2-2. Generalized Integration

In order to understand better the operation of the Electronic Generalized Integrator, we should briefly examine the definition of a definite integral. This is expressed in Equation (2-1), where the function f(X) is defined to be continuous over the interval between a and b.

$$\int_a^b f(X)\,dX = \lim \sum_{i=1}^{n} f(X_i') \,\Delta X_i \tag{2-1}$$

where

$$\Delta X_i = X_i - X_{i-1}$$

and

$$X_{i-1} \leq X_i' \leq X_i$$

The limit is evaluated as n is increased and as the maximum of the numbers $\Delta X_i, \ldots \Delta X_n$ is made to approach zero. A geometrical interpretation of Equation (2-1) is illustrated in Figure 1. If we now define $\Delta X$ to be a constant and make this value very small, Equation (2-1) may be replaced by the following approximation:

$$\int_a^b f(X)\,dX \approx \Delta X \sum_{i=1}^{n} f(X_i') \quad . \tag{2-2}$$

This approximation describes the integral as the summation of a number of

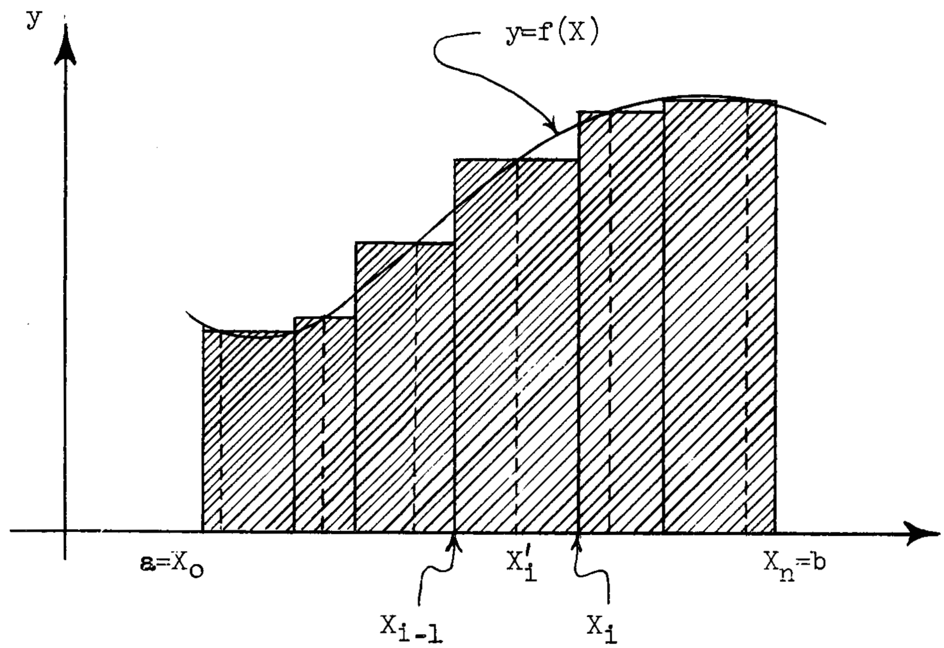


Figure 1: Geometrical Interpretation of a Definite Integral

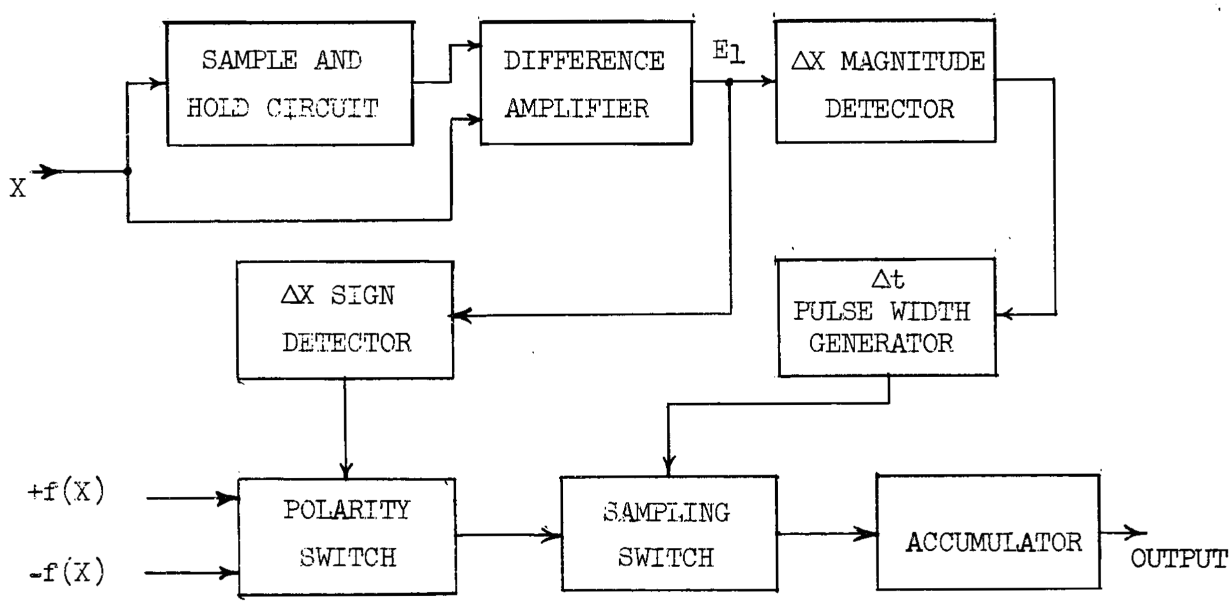


Figure 2: Block Diagram of Analog Program Used for Generalized Integration

samples of f(X) times the constant ΔX. Since X and f(X) will both be functions of time, we may make the substitution of a sampling width of Δt seconds for the ΔX appearing in Equation (2-2). This substitution will require that the length of the time Δt for which the sample is taken be less than the shortest time required for a change in the variable X of amount ΔX.

The instrumentation of Equation (2-2) was described by Bekey in 1958*. A simplified block diagram of the analog program he used to perform this instrumentation is illustrated in Figure 2. In this program, the variable X is examined for changes of a preset amount which is defined to be ΔX. These changes are detected by storing the value of the X input at some time $t_o$ in the Sample and Hold Circuit and comparing this value with the value of X at some later time $t_1$. The comparison is made by the Difference Amplifier and the output of this section may be described as in Equation (2-3):

$$E_1 = K[X(t_1) - X(t_o)] \quad (2\text{-}3)$$

When the magnitude of this value reaches the predetermined level, an output signal is generated in the ΔX Detector. This signal causes a sample of f(X) that is Δt seconds long to appear at the input of the Accumulator. The time, $t_o$, at which the Sample and Hold Circuit is reset to the value of the variable X is the instant just after that when f(X) is sampled. Since the change in the X input may be either positive or negative, the output of the Difference Amplifier is simultaneously examined for its polarity by the Sign Detector. The Sign Detector controls the polarity of the f(X) samples that appear at the input of the Accumulator. If the incremental change is negative, the negative of f(X) is sampled. The output of the Accumulator then contains the summation of all the input samples for X on the interval between a and b. If the value of ΔX that is preset into the ΔX Detector is sufficiently small when compared to the difference between the two limits of integration, then the output of the Accumulator is a good approximation to the integral as expressed in Equation (2-2).

## 2-3 . The Electronic Generalized Integrator

The Electronic Generalized Integrator constitutes an instrumentation of Equation (2-2) which is similar in many respects to that used by Bekey. It

*Bekey, George A., "Generalized Integration on the Analog Computer", presented at the National Simulation Conference, Dallas, Texas, 23-25 October 1958.

differs primarily in the method whereby the incremental changes in the X input are detected and in the use of electronic switches for the sampling gates. The basic block diagram is shown in Figure 3.

The integrator may be divided into two separate parts, the first being the section used for the detection of incremental changes in the X input. This Incremental Detector section is a simplified version of the AID Converter*. The information derived from this section may be considered an incremental digital representation of the X input. When the input to this section changes by a prechosen amount, the output signal relays this event and information on the sign of this change to the sampling circuits contained in the other section of the integrator.

The Incremental Detector is a closed-loop circuit that uses an operational amplifier with capacitive feedback for the analog storage element. The inputs to this amplifier are the Reset Pulses and the analog variable X. The analog input to the amplifier is coupled through a series capacitor which results in the analog signal appearing at the output of this amplifier with a sign inversion and a suitable scale factor. The Reset Pulses are introduced to the grid of this amplifier through a series resistor which results in the integration of these pulses. The volt-second area of the Reset Pulses and the integrating gain of the operational amplifier circuit are so chosen that the change in amplifier output voltage resulting from one Reset Pulse is equal to that produced by a change in the input analog signal of an amount equal to $\Delta X$. The output voltage of this amplifier is the error voltage of the closed loop circuit. When it exceeds an amount equal to $\Delta X$, this event is sensed by the Threshold Detector. The output of the Threshold Detector is used to generate a Reset Pulse and to close the sampling gate when this event occurs. The polarity of the Reset Pulse is selected to reduce the magnitude of the voltage at the output of the operational amplifier providing negative feedback in the closed-loop circuit of the Incremental Detector section. An illustration of typical waveforms is given in Figure 4.

The second portion of the Electronic Generalized Integrator has circuitry similar to that described in Section 2-2. Electronic switches are used for the

*"New Methods and Application of Analog Computation", Final Summary Report on Contract NO. DA-01-009 ORD-853, GIT/EES Report A497/T1, 15 may 1961.

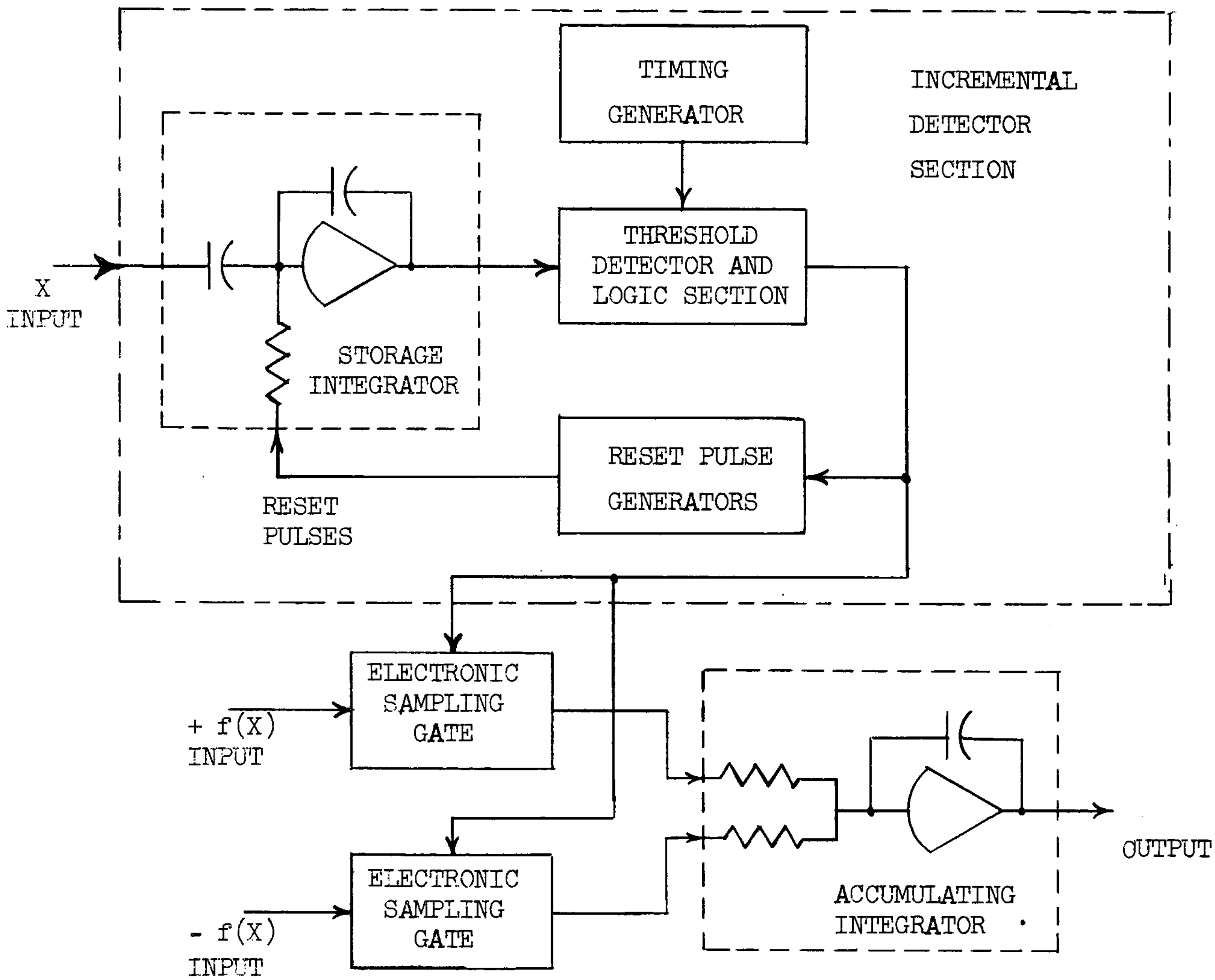


Figure 3: Basic Block Diagram of the Electronic Generalized Integrator

sampling gates in order to allow the sampling time of the variable f(X) to be decreased. This decrease is necessary in order to increase the bandwidth of the Generalized Integrator. The accumulation of the samples of f(X) is accomplished with an operational amplifier with capacitive feedback and resistive input. The output voltage of this amplifier changes in a staircase manner, with each voltage step being proportional to the amplitude of the corresponding sample of f(X). This effect is illustrated in Figure 4. For the output of the Electronic Generalized Integrator to be a useful representation of the integral of f(X), the size of $\Delta X$ must be made small so that the curve pictured in Figure 4 approaches a smooth curve.

## 2-4. The Incremental Detector

It was pointed out in the previous section that the Incremental Detector of the Electronic Generalized Integrator is a simplified version of the Aid Converter. The detailed block diagram of this section is shown in Figure 5. Since the need for digital readout circuits has been eliminated, a simplification in the logic circuit may be made to reduce the total number of stages required. The other simplification is based on the elimination of the absolute-value circuit from the analog program. This was done in an effort to avoid some of the difficulties experienced in the frequency response in this section of the AID Converter. The absolute value circuit has been replaced by separate positive and negative threshold detectors. The threshold detector circuit is identical to that used in the AID Converter. It is basically a NOR circuit in which the condition at the output is determined by both of the input signals. The circuit is shown in Figure 6. One input signal is a square wave that is supplied as an interrogation pulse to both detectors. If and only if the error voltage is above the threshold level of a detector during the interval of the interrogation pulse, a pulse appears at the output of this detector. The Positive Threshold Detector is a complementary circuit of the Negative Threshold Detector. That is, the NPN transistor is replaced by a PNP transistor and all voltages are reversed in sign. It then performs identically to the Negative Threshold Detector for error voltages of the opposite polarity. A pulse from the output of one detector indicates that the X variable has changed by an amount equal to $\Delta X$ in the positive direction, while a pulse from the output of the other detector indicates that the X variable has changed by an amount equal to $\Delta X$ in the negative direc-

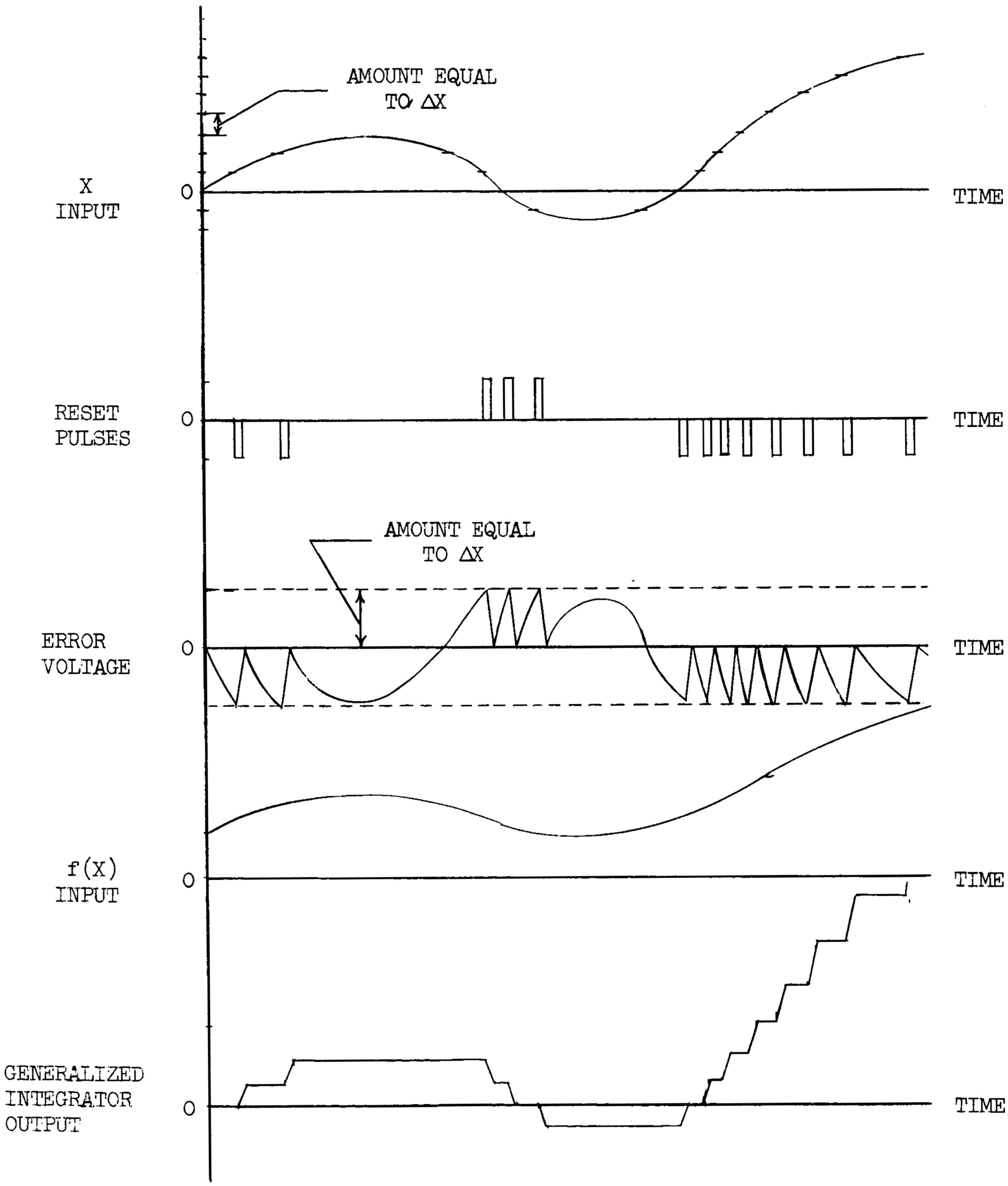


Figure 4: Typical Waveforms Found in the Electronic Generalized Integrator

tion. The reset pulse then reduces the magnitude of the error voltage below the threshold value.

A pulse occuring at the output of a given Threshold Detector is amplified and used to set a flip-flop circuit that is associated with that detector. A pulse from a timing source contained in the Electronic Generalized Integrator is applied simultaneously to the reset inputs of both flip-flop circuits. This pulse follows the Interrogation Pulse after an accurately controlled interval. This arrangement allows the output pulses of the flip-flops to be used to control the length of the Reset Pulses in the Incremental Detector and the length of the f(X) sampling pulses appearing at the input of the Accumulating Integrator.

## 2-5. The Electronic Sampling Gate

The Sampling Gate is a transformer-driven switching circuit using two transistors as shown in the circuit diagram of Figure 7. The transformer connections to the transistors are so phased that an input signal into the primary of the transformer turns on both transistors and causes them to saturate. In the absence of this input signal, no base current is supplied to the transistors and they may be considered in a cutoff state. This condition is equivalent to having a very large resistance between the input and the output terminals of the gate circuit. When the pulse is applied to the input of the transformer the two transistors saturate, reducing the equivalent resistance of the gate circuit by a factor of approximately a million. Since the two transistors are connected with their emitters together, they form a bidirection switch and will accept signals of both polarities. The time required for the gate to change states may be made small compared to the time the gate is closed. Errors due to the finite resistance in the gate circuit in the open condition are somewhat compensated by the second gate circuit, which is connected to a voltage that is equal in magnitude but opposite in sign to that connected to the first gate. If this compensation is not complete, the remainder of the analog signal may be nulled out. The resistance in the gate circuit remaining in the on period is in series with the resistor to the grid of the Accumulating Amplifier and is accounted for in adjusting the gain of this circuit.

## 2-6. The Analog Program

The analog program included in the Electronic Generalized Integrator is illustrated in Figure 8. Two operational amplifiers are being included in the

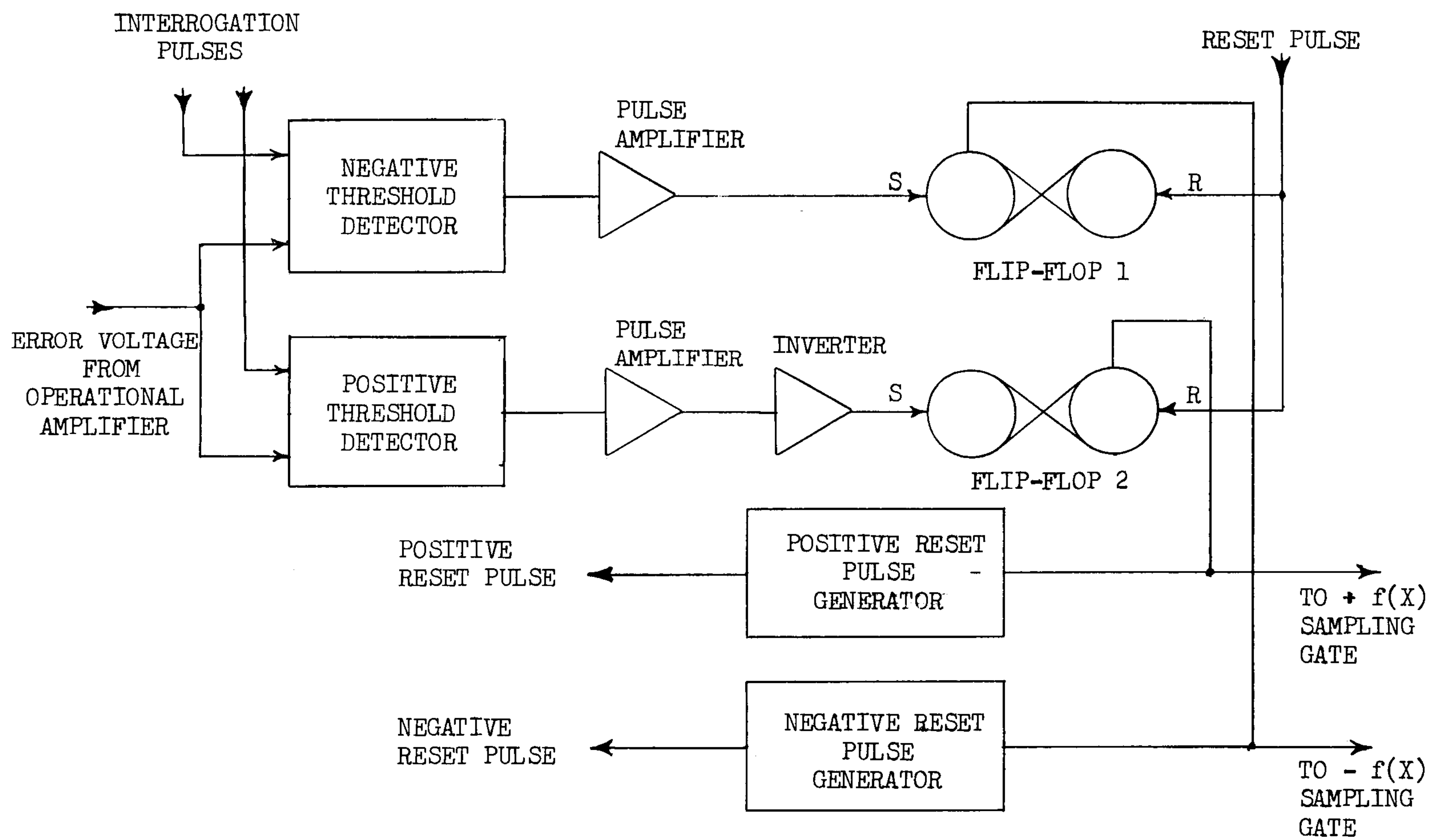


Figure 5: Block Diagram of the Threshold Detectors and Logic Section

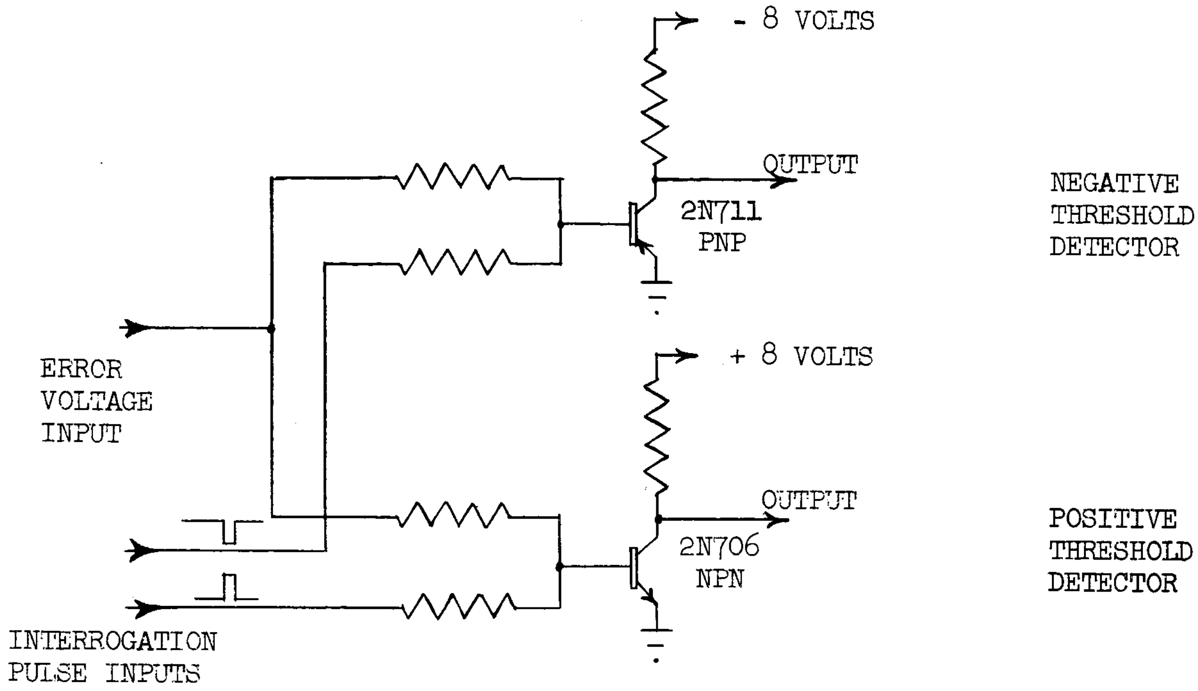


Figure 6: Circuit Diagram of the Threshold Detectors

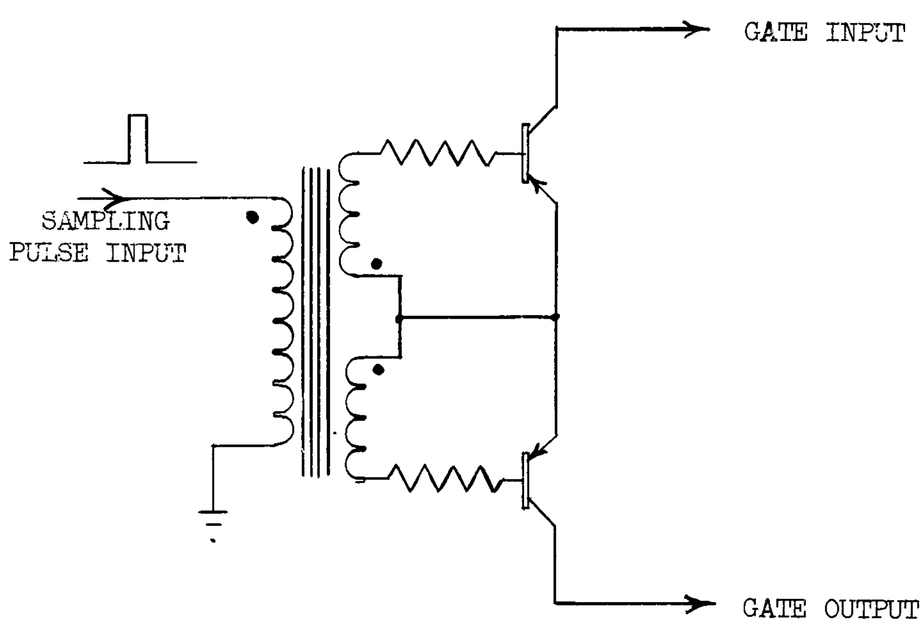


Figure 7: Circuit Diagram of the Electronic Sampling Gates

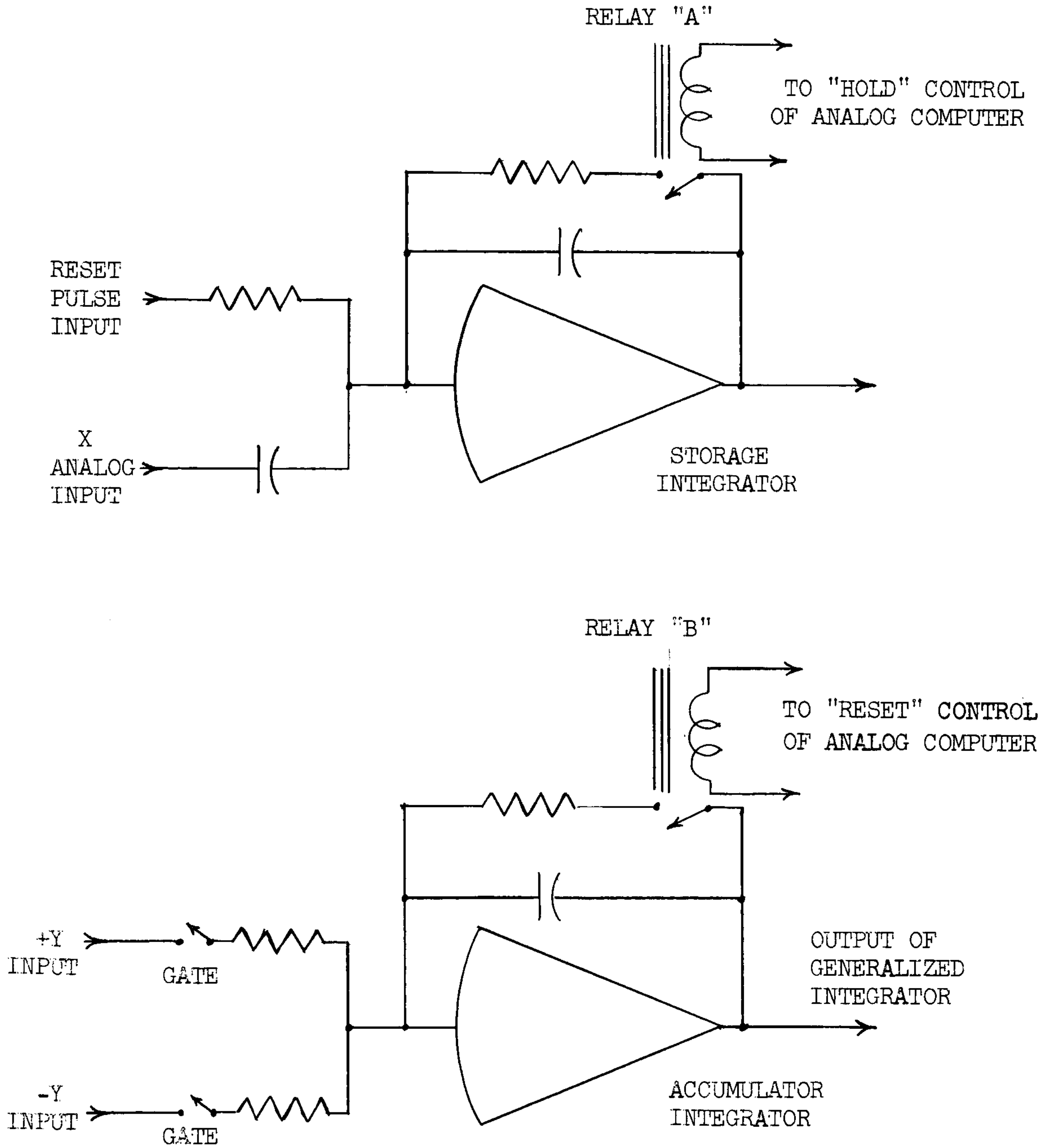


Figure 8: Analog Program of the Electronic Generalized Integrator

initial design. If drift problems in the Storage Integrator contained in the Incremental Detector section become significant, it may be necessary to share the amplification in this stage over two amplifiers. Two relays are included in the circuits to provide a means of slaving the Electronic Generalized Integrator to the controls of an analog computer. The relay in the circuit of the Accumulator Integrator is used to bring the output of this amplifier to zero, which resets the Electronic Generalized Integrator. This relay is intended to be controlled by the RESET control on the analog computer. The relay in the circuit of the Storage Integrator of the Incremental Detector is used to place the Electronic Generalized Integrator in a HOLD condition. This relay resets the Storage Integrator, holding the output voltage of this amplifier at zero, but not disturbing the voltage stored in the Accumulator Integrator.

## 2-7. Work Completed

The major portion of the design of the Electronic Generalized Integrator has been completed and the parts necessary for construction of a working model have been placed on order. It is intended that this working model be a complete, self-contained unit capable of being used with a standard general-purpose analog computer.

Construction has begun on a minor portion of the circuit, but this work has not progressed very far due to the lack of parts. A breadboard of the circuit used in the Sampling Gate has been built and preliminary evaluation of this circuit has shown that it will operate with a linearity of better than 0.2 percent over the intended full-scale range. Better performance is expected by adjusting the operating point of the gate transistors. If still further improvement is needed, matched transistors will be used.

## 2-8. Future Work

The work during the next quarter should include the completion of a working unit of the Electronic Generalized Integrator and a preliminary evaluation of its performance.

## Chapter III

## GENERATION OF NONSTATIONARY NOISE FOR ANALOG-COMPUTER MONTE CARLO STUDIES

Task II of GIT/EES Project A-588 concerns the investigation of analog techniques for the generation of nonstationary noise voltages to be used in analog Monte Carlo studies. Of particular interest is the problem of nonstationary shaping of Gaussian, band-limited, white noise. As discussed in the Sections below, efforts during the past quarter have centered about three principal topics--analysis of certain specific stochastic processes, selection of statistical functionals useful in specifying nonstationary noise, and investigation of linear networks for nonstationary shaping.

Current plans are to issue separately a project technical note covering the fundamentals of stochastic process theory as it pertains to Task II. This note will serve as a project reference manual and will define several functionals which are inconsistently defined in the open literature.

### 3-1. Analysis of Certain Stochastic Processes

It is not possible to exhibit a convenient representation of the arbitrary stochastic process. It is instructive, however, to examine the statistics of a few "closed-form" processes which are expressible as time functions with random parameters.

#### a. Sinewave of Random Phase and Amplitude

The stochastic process represented by

$$X(t) = x \sin(t+y) ,$$

where x and y are independent random variates, might arise through the use of a sinewave generator whose output amplitude varies slowly in a random way so that for each sample function the amplitude may be considered a constant. The random variate y takes into account the random phase of the signal with respect to the beginning of the sample. (To make this process more nearly an exact representation of the experiment, the interval between sample functions should be sufficiently long to provide independence.)

$$\text{(3-1)} \qquad EX(t) = Ex\, E\sin(t+y) = (\sin t)\, Ex\, E(\cos y) + (\cos t)\, Ex\, E(\sin y) ,$$

which, in general, is a function of t.

The covariance function is given by

$$\text{(3-2)} \qquad Cov(X) = C_X(T,t) = EX(t)X(t+T) = E[x^2 \sin(t+y) \sin(t+T+y)]$$

$$= \frac{1}{2} Ex^2 [\cos T - \cos(2t+T)\, E\cos 2y + \sin(2t+T)\, E\sin 2y] ,$$

which, in general, is also a function of time, t.

The time average of X(t) is given by

$$(3\text{-}3) \qquad AX(t) = \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{a} X(t)\, dt = \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{a} x \sin(t+y)\, dt = 0,$$

invariant with respect to the sample function chosen.*

The time-statistic equivalent of the covariance function, the correlation function, is

$$R_X(T,x,y) = AX(t)X(t+T) = \lim_{a\to\infty} \frac{1}{2a} \int_{-a}^{a} x^2 \sin(t+y)\, \sin(t+y+T)\, dt ,$$

which reduces to a form invariant with respect to y:

$$(3\text{-}4) \qquad R_X(T,x) = \frac{1}{2}\, x^2 \cos T.$$

If we assume that the random phase, y, is uniformly distributed over the interval $(-\pi,+\pi)$ with probability density function

$$(3\text{-}5) \qquad \begin{aligned} p_y(y) &= \frac{1}{2\pi}, \quad |y| < \pi \\ p_y(y) &= 0 , \quad |y| \geq \pi \end{aligned}$$

then the process mean, as given by Equation (3-1), reduces to zero identically in t. That is,

$$EX(t) = AX(t) = 0, \text{ all } t.$$

Also, under the assumption of (3-5), the covariance becomes invariant in t and Equation (3-2) reduces to

$$(3\text{-}6) \qquad C_X(T) = \frac{1}{2}(\cos T)\, Ex^2 ,$$

which is characteristic of the "wide-sense stationary" process.

*The symbol "A" is used herein to denote the time-average operator, as contrasted with the ensemble-average operator "E".

If the power spectral density function, $f_X$, is defined as the Fourier transform (with respect to T) of the covariance function,*

(3-7) $$f_X(\omega,t) = \int_{-\infty}^{\infty} C_X(T,t)\, \exp(-j\omega T)\, dT \quad ,$$

then Equation (3-6) yields

(3-8) $$f_X(\omega) = \frac{\pi}{2}\, Ex^2\, [\delta(\omega+1) + \delta(\omega-1)] \quad .$$

Note that the definition of (3-7) applied to the nonstationary process of Equation (3-2) yields a complex power spectral density function which is of little physical significance. Other definitions in common use, such as those suggested by Page** and Middleton***, have other drawbacks which limit their usefulness. Page's "instantaneous power spectrum" yields a random variate and Middleton's "intensity density" is identically zero for all finite-energy processes. Considerable attention is given to this problem in the forthcoming technical note.

b. Sinewave of Random Phase and Frequency

Let x and y be independent random variates with the probability density function of y given by

$$p_y(y) = \frac{1}{2\pi} \quad , \quad |y| \leq \pi$$

$$p_y(y) = 0 \quad , \quad |y| > \pi$$

and that of x satisfying

(3-9) $$p_x(x) = p_x(-x) \quad .$$

*There does not appear to be a consistent definition of the power spectral density function for nonstationary processes. The definition used here is essentially that of Kharkevich: A.A. Kharkevich, Spectra and Analysis, Consultants Bureau, New York, 1960 (Translated from Russian), p. 149.

**C.H. Page, "Instantaneous Power Spectra", Journal of Applied Physics, Vol. 23, No. 1, Jan. 1952.

***David Middleton, An Introduction to Statistical Communication Theory, McGraw-Hill Book Company, New York, 1960.

Construct the stochastic process

$$X(t) = 2\sin(xt + y) ,$$

where the factor "2" is chosen (without loss of generality) to make the average power unity. This process would arise, for example, through the use of a sine-wave generator whose frequency varies slowly in a random way so that for each sample function the frequency may be considered a constant. As in Section 3-1a, variate y accounts for the random phase of the signal with respect to the origin.

The expected value of X is zero for all t and the covariance function is invariant in t:

$$C_X(T) = E(\cos xT) = \int_{-\infty}^{\infty} p_x(x) \cos xT \, dx \quad .$$

The process is therefore wide-sense stationary and the definition of power spectral density given by Equation (3-7) is applicable. Applying this definition, we obtain

$$f_X(\omega) = \int_{-\infty}^{\infty} C_X(T) \exp(-j\omega T) \, dT = \int_{-\infty}^{\infty} \exp(-j\omega T) \int_{-\infty}^{\infty} p_X(x) \cos xT \, dx \, dT \, .$$

Interchanging the order of integration,

$$f_X(\omega) = \int_{-\infty}^{\infty} p_X(x) \int_{-\infty}^{\infty} \exp(-j\omega T) \cos xT \, dT \, dx$$

$$= \pi\int_{-\infty}^{\infty} p_x(x) \, [\delta(\omega-x) + \delta(w+x)] \, dx$$

$$= \pi \, [p_x(\omega) + p_x(-\omega)] \quad .$$

From Equation (3-9)we may write, finally,

$$f_X(\omega) = 2\pi p_X(\omega) \, .$$

Thus, this process has the unusual property that the power spectral density (in the sense of Equation (3-7)) is proportional to the probability density of the principal primitive variate.

c. Gaussian, Band-Limited, White Noise

Most commercially available noise generators deliver an output voltage which approximates the Gaussian, band-limited, white noise process. It is useful, therefore, to express this process in closed form. Such a representation can be developed as shown below.

It is convenient to discuss first the following lemma:

Lemma I: Let $S = \sum_{n=-\infty}^{\infty} \frac{\sin(u-n\pi)}{u-n\pi} \cdot \frac{\sin(u+v-n\pi)}{u+v-n\pi}$

then $S = \frac{\sin v}{v}$.

Proof: Write S in the form

$$S = \sum_{n=-\infty}^{\infty} \frac{\sin u \cos n\pi}{u-n\pi} \cdot \frac{\sin(u+v) \cos n\pi}{u+v-n\pi}$$

$$= \sin u \sin(u+v) \sum_n \frac{1}{(u-n\pi)(u+v-n\pi)}$$

Expand the summand by partial fractions:

$$S = \sin u \sin(u+v) \sum_n [\frac{1/v}{(u-n\pi)} - \frac{1/v}{(u+v-n\pi)}]$$

$$= \frac{\sin u \sin(u+v)}{v} \sum_n [\frac{1}{u-n\pi} - \frac{1}{u+v-n\pi}].$$

Now

$$\sum_n \frac{1}{u-n\pi} = \frac{1}{u} + \sum_{n\neq 0} \frac{1}{u-n\pi} = \text{ctn } u - \sum_{n\neq 0} \frac{1}{n\pi}$$

and

$$\sum_n \frac{1}{u+v-n\pi} = \text{ctn}(u+v) - \sum_{n\neq 0} \frac{1}{n\pi}.^*$$

*See, for example, D.H. Menzel, Fundamental Formulas of Physics, Section 14.8, Dover Publications, New York, 1960.

Then S may be expressed as

$$S = \frac{\sin u \sin(u+v)}{v} [\operatorname{ctn} u - \sum_{n\neq 0} \frac{1}{n\pi} - \operatorname{ctn}(u+v) + \sum_{n\neq 0} \frac{1}{n\pi}]$$

$$= \frac{\sin u \sin(u+v)}{v} \cdot \frac{\sin v}{\sin u \sin(u+v)} .$$

And, finally,

$$S = \frac{\sin v}{v} , \text{ QED.}$$

For the construction of the Gaussian, band-limited, white noise process, let $x_n$ be the generic symbol for a set of independent, zero-mean, Gaussian random variates with common variance, $\sigma^2$. Form the stochastic process, $X_N$,

$$X_N = \sum_{n=-N}^{N} x_n \frac{\sin(at-n\pi)}{at-n\pi} .$$

Then $X_N(t_1),\ldots,X_N(t_i),\ldots,X_N(t_k)$ is a k-variate Gaussian variable for every N. Since

$$Ex_i x_j = 0, \ i\neq j,$$

and

$$Ex_i^2 = \sigma^2 , \text{ all } i,$$

and

$$\sum_{n=-\infty}^{+\infty} [\frac{\sin(at-n\pi)}{at-n\pi}]^2 = 1 \quad \text{(a special case of Lemma I)} ,$$

then there exists a limiting random variable, $X(t)$, given by

$$X(t) = \lim_{N\to\infty} X_N(t) = \sum_{n=-\infty}^{\infty} x_n \frac{\sin(at-n\pi)}{at-n\pi} .$$

Further, $X(t)$ is Gaussian with zero mean and variance $\sigma^2$. The covariance of X is given by

$$C_X(T) = EX(t)X(t+T)$$

$$= E\{[\sum_n x_n \frac{\sin(at-n\pi)}{at-n\pi}] [\sum_n x_n \frac{\sin(at+aT-n\pi)}{at+aT-n\pi}]\} .$$

Because of the independence of the $x_n$'s, this may be written

$$C_X(T) = E \sum_n x_n^2 \frac{\sin(at-n\pi)}{at-n\pi} \cdot \frac{\sin(at+aT-n\pi)}{at+aT-n\pi} = \sigma^2 \sum_n \frac{\sin(at-n\pi)}{at-n\pi} \cdot \frac{\sin(at+aT-n\pi)}{at+aT-n\pi}.$$

By Lemma I, this reduces to

$$C_X(T) = \sigma^2 \frac{\sin aT}{aT},$$

which is invariant in t and, therefore, represents a wide-sense stationary process. But X(t) has been shown to be a Gaussian process and is then strictly stationary.*

The power spectral density may now be computed as the Fourier transform of $C_X$:

$$f_X(\omega) = \frac{\pi\sigma^2}{a}, \quad |\omega|<a$$

$$f_X(\omega) = 0, \quad |\omega|\geq a.$$

Hence, the stochastic process represented by

(3-10) $$X(t) = \sum_{n=-\infty}^{\infty} x_n \frac{\sin(at-n\pi)}{at-n\pi},$$

where the $x_n$'s are independent Gaussian variates with zero mean and common variance, is a stationary Gaussian band-limited white noise process with bandwidth a.

## 3-2. Functionals Useful in Specifying Nonstationary Noise

The specification of statistical properties in stationary processes is not a particularly difficult proposition. In a given Monte Carlo study, for example, we might require a noise souce which delivers a stationary process having a specified power spectral density (or, equivalently, a certain covariance function) and a specified univariate probability distribution. It may be difficult, in cases, to generate the desired process, but it is generally not hard to determine what is required.

*J.H. Laning and R. H. Battin, Random Processes in Automatic Control, McGraw-Hill Book Company, New York, 1956, p. 156.

The selection of appropriate functionals to describe a nonstationary process, on the other hand, may require much effort. In the first place, several of the functionals commonly used are not universally defined in the same way. Secondly, the nonstationarity rules out the possibility of ergodicity and requires that meaningful specifications be made on the ensemble basis.

Considerable effort has been devoted to this problem during the past quarter, with the following tentative conclusions:

(a) The covariance function, as defined by Equation (3-2), appears to be a more useful and meaningful function than the power spectral density. As mentioned in Section 3-1, this latter quantity is not well defined and may lead to a complex functional or a random variate.

(b) Generally speaking, the complete specification of a nonstationary process is impractical. If $X(t)$ represents the process, a complete description is provided only by citing the k-variate distribution of $[X(t_1),\ldots,X(t_k)]$ for every integral k (or the equivalent information). Thus it is necessary to select carefully the statistical parameters of principal interest. (This is true of many stationary processes, also, but appears to be a more vital question in the nonstationary case.)

With regard to the input random process, two decisions must be made when setting up a Monte Carlo study: (1) Which functionals or parameters will be used to describe the random process, and (2) What is the detailed form of the chosen functionals. Decision (1) is usually made on the basis of the response variables of interest in the system under study. Decision (2) is dictated by the physical nature of the stochastic process being simulated. To illustrate the way in which these latter specifications are derived, we consider the following very general example.

A rocket is to be fired in a vertical attitude and assumed to rise at a constant rate. As it moves through the atmosphere, it is subject to buffeting winds which are random in nature. It is desired to simulate the rocket system and observe its response to the simulated random wind. A number of computational runs will be performed to gather statistical data on the rocket performance.

Conceptually, at least, the required data on the statistical behavior of the wind could be obtained by placing sensing elements at various altitudes and recording sample functions of wind velocity. If we assume that the speed and direction of the wind are independent scalar quantities, then these may be simulated separately and combined in the proper relationship by a resolver. For simplicity we will consider only the scalar wind speed.

Viewed as a whole, the wind speed may be thought of as a stochastic process of two parameters--the altitude, h, and time, t. Let the process be denoted by X(h,t) and symbolize the mean and covariance by $\mu_X(h,t)$ and $C_X(h,H,t,T)$:

$$\mu_X(h,t) = EX(h,t)$$

$$C_X(h,H,t,T) = EX(h,t)X(h+H,t+T)$$

If it is understood that the rocket will be launched only under good weather conditions and at a particular time of day, then it may be assumed that the wind at a given altitude may be represented by a stationary stochastic process.* Under these conditions, the mean and covariance are invariant in t and may be written as $\mu_X(h)$ and $C_X(h,H,T)$.

Since the rocket is to rise at a constant rate, v, we have for the altitude

$$h = vt$$

and

$$H = vT.$$

The process may now be written in terms of a single parameter, t: X(vt,t). This is expressed more concisely as a new process Y(t) having mean and covariance given by

$$\mu_Y(t) = EY(t) = \mu_X(vt) = EX(vt,t)$$

$$C_Y(T,t) = EY(t)Y(t+T) = C_X(vt,vT,T) = EX(vt,t)X[v(t+T),t+T] \quad .$$

If these quantities seem appropriate to test that portion of the rocket in which we have an interest, then efforts may be made to generate a nonstationary process having these properties. Otherwise, more appropriate functionals may be derived in the same way and attempts made to generate a suitable process.

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*However, the rocket will be changing altitude constantly and this consideration will lead to a nonstationary process.

## 3-3. Linear Networks for Nonstationary Shaping

There appear to be at least three techniques which may be employed (separately or in combination) to derive nonstationary stochastic processes having prescribed properties: nonlinear networks (possibly time-varying), time varying linear networks, and utilization of transient effects in time-invariant networks. During the past quarter attention has been given primarily to the use of linear networks.* The results are summarized below.

### a. Time-Varying Linear Networks

The analysis of time-varying networks has been extensively studied by several authorities, notably Zadeh and Darlington. Zadeh in particular has developed techniques for obtaining the system function and impulsive response of linear variable networks from the so-called fundamental equation of the network---i.e., the differential equation which relates the input and output. If the fundamental equation of the network is of the form

$$L(p,t)y(t) = K(p,t)x(t)$$

where $x(t)$ and $y(t)$ are the input and output of the network and L and K are linear differential operators, then the system function, $H(j\omega,t)$, for the network must satisfy the following differential equation:

$$\frac{1}{n!L}\frac{\partial^n L}{\partial(j\omega)^n}\frac{d^n H}{dt^n} + - - - + \frac{1}{L}\frac{\partial L}{\partial(jw)}\frac{dH}{dt} + H = \frac{K}{L} \quad . \; ^{**}$$

The boundary conditions for the above equation must either be given or be derivable from the network. In practice, even for relatively simple networks, the equation for $H(j\omega,t)$ is so complex as to preclude the possibility of obtaining a closed-form solution and requires the use of some rather sophisticated

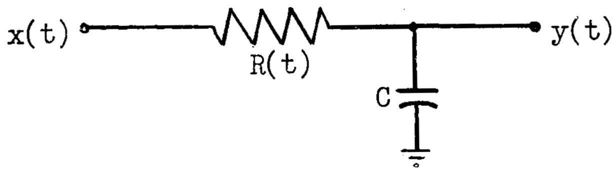
---

*One rather interesting nonlinear technique has been discovered. Let a diode function generator be driven by a noise source having probability distribution $P(x)$. Adjust the function generator to the functional form of $P(x)$. Then the function generator output will have probability density uniform on the interval (0,1) and zero elsewhere. This technique follows from the theorem: "Any density for a continuous variate x may be transformed to the uniform density $f(y) = 1 \quad (0 < y < 1)$ by letting $y = G(x)$, where $G(x)$ is the cumulative distribution of x." (A.F. Mood, Introduction to the Theory of Statistics, McGraw-Hill Book Company, New York, 1950, page 107.

**L.A. Zadeh, "Frequency analysis of variable networks," Proc. IRE, vol. 38, pp. 291-299, March, 1950.

approximation techniques which are, at best, applicable only over a limited range of the independent variable.

As an example of the techniques developed by Zadeh, consider the following circuit.




It is a simple matter to write the fundamental equation for this circuit:

$$CR(t)\frac{dy(t)}{dt} + y(t) = x(t) \quad , \text{ or}$$

$$[CR(t)p + 1]\, y(t) = x(t) \quad , \qquad p = \frac{d}{dt} \quad .$$

The system function is then obtainable from the derived equation

$$\frac{dH}{dt} + \frac{j\omega CR(t) + 1}{CR(t)}\, H = \frac{1}{CR(t)} \quad ,$$

which yields

$$H(j\omega, t) = \exp\left[-\int_0^t \frac{j\omega CR(\alpha) + 1}{CR(\alpha)}\, d\alpha\right] \int_0^t \frac{1}{CR(\beta)} \exp\left[\int_0^\beta \frac{j\omega CR(\gamma) + 1}{CR(\gamma)}\, d\gamma\right] d\beta \quad .$$

Having obtained the system function for the network, it is now possible to derive the impulsive response — i.e., the response of the network at time t to an impulse applied at time $t_o$. Zadeh shows that the impulsive response $W(t,t_o)$ and the system function are related by the following integral equation:

$$W(t,t_o) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(j\omega, t)\, e^{j\omega(t-t_o)}\, d\omega \quad , \qquad t > t_o \quad ,$$

which yields for $W(t,t_o)$

$$W(t,t_o) = \frac{1}{CR(t_o)} \exp\left[-\int_{t_o}^{t} \frac{d\alpha}{CR(\alpha)}\right] \quad , \qquad t > t_o \quad .$$

Notice that when R(t) is constant, the results obtained above reduce to the system function and impulsive response for the simple R-C section low-pass filter as expected:

$$H(j\omega,t) = H(j\omega) = \frac{1}{jCR\omega + 1}$$

$$W(t,t_o) = W(t-t_o) = \frac{1}{RC} e^{-(t-t_o)/RC} , \quad t > t_o .$$

Suppose the network in the example is driven by band-limited Gaussian white noise such that the input power spectral density function is given by

$$G_1(\omega) = \begin{cases} b, & |\omega| \le a \\ 0, & \text{otherwise} \end{cases}$$

and the corresponding covariance function is given by

$$C_1(T) = E[x(t)x(t+T)] = \frac{ab}{\pi} \frac{\sin(aT)}{aT} .$$

Since the input has zero mean, the output also has zero mean; further, the covariance function for the output process is given by

$$C_2(t,T) = E[y(t)y(t+T)]$$

$$= \frac{ab}{\pi} \exp\left[-2\int_0^t \frac{d\alpha}{CR(\alpha)} - \int_t^{t+T} \frac{d\alpha}{CR(\alpha)}\right] \int_0^t \int_0^{t+T} \frac{\dfrac{\sin a(\beta-\gamma)}{a(\beta-\gamma)}}{C^2 R(\beta) R(\gamma)} \exp\left[\int_0^\beta \frac{d\alpha}{CR(\alpha)} + \int_0^\gamma \frac{d\alpha}{CR(\alpha)}\right] d\beta\, d\gamma .$$

No attempt will be made here to derive an output power spectral density function. It may be noted in this connection, however, that Fourier transformation of the output covariance function yields a complex expression with little physical significance.

Consider now a particular form for R(t):

$$R(t) = \frac{t}{C} .$$

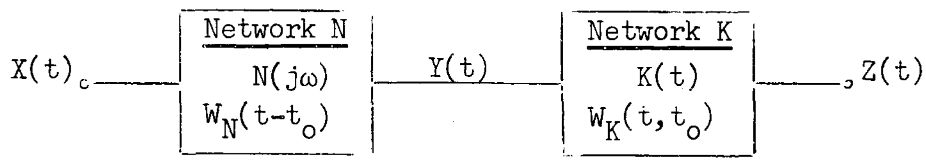
Substituting this expression for R(t) back into the previous results yields

$$H(j\omega,t) = \frac{1}{j\omega t}\left[1 - e^{-j\omega t}\right] ,$$

$$W(t,t_o) = \frac{1}{t} ,$$

$$C_2(t,T) = \frac{ab}{\pi t(t+T)} \int_0^t \int_0^{t+T} \frac{\sin a(\beta-\gamma)}{a(\beta-\gamma)} d\beta\, d\gamma .$$

It is clear from the example and from the general equation for the system function of a variable network that even relatively simple networks having only one variable element lead to expressions for network parameters which are so cumbersome as to be all but unmanageable. Since the analysis of known networks lead to such unwieldly expressions, it is easy to see that the synthesis of such networks borders on the impossible. Because the synthesis problem is of primary importance, the class of variable networks under consideration has been restricted to the so-called separated networks. These are variable networks which can be represented by the cascaded combination of a linear invariant network and a linear variable network that contains no reactive components. The block diagram for a typical separated network is shown below.




The behavior of the network N is well known; further, it is easy to show that since $Z(t) = K(t)Y(t)$, then $E[Z(t)] = K(t)E[Y(t)]$ and the covariance function for $Z(t)$ is given by

$$C_Z(t,T) = E[Z(t)Z(t+T)] = K(t)K(t+T)\ C_Y(T) \quad .$$

It is clear that this class of variable networks yields more tractable results than does the general variable network, while not seriously compromising either flexibility or utility. The properties of this general class and applicable synthesis techniques will be more extensively investigated during the coming report period.

b. Invariant Linear Networks

The technique of utilizing the transient response of time-invariant networks to derive nonstationary stochastic processes has been investigated by Lampard.* The covariance function and probability distribution function of the output process are derived when the input is stationary, white, and Gaussian, but no attempt is made to develop synthesis techniques.

*D.G. Lampard, "The response of linear networks to suddenly applied stationary random noise," IRE Trans. on Circuit Theory, Vol. CT-2, 49-57, March 1955.

It is felt that this technique and the results derived by Lampard hold a great deal of promise for that restricted class of problems where it is sufficient to specify only a single parameter of the output process. As time permits beyond the work outlined in the preceding subsection, this technique will be further investigated in light of its application to the problem of interest and an attempt will be made to develop procedures for synthesis.

## 3-4. Planned Work for the Coming Quarter

During the coming report period, efforts on Task II will be directed toward completion of the technical note discussed in the introductory paragraph of Chapter III and toward further development of the linear-network shaping techniques. If time permits, a limited amount of experimental investigation will be conducted using the Georgia Tech Analog Computer Laboratory facilities. The principal aim of this empirical work will be verification of the shaping techniques to be developed, particularly those resulting from the "separated networks" discussed in Subsection 3-3a.